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Symmetry analysis of the two-dimensional diffusion equation with a source term

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Dedicated to William Ames on his 80th birthday

Abstract

A symmetry analysis is performed on a $(2+1)$-dimensional linear diffusion equation with a nonlinear source term involving the dependent variable and its spatial derivatives. In the first part of the paper, we use the classical method to classify source terms where the original equation admits a nontrivial symmetry. In the second part of the paper, we use the nonclassical method and show that we simply recover the classical symmetries.

Keywords: Classical and nonclassical symmetry; Diffusion equation; Nonlinear source term

1. Introduction

Symmetry analysis has played a fundamental role in the construction of exact solutions to nonlinear partial differential equations. Based on the original work of Lie [1] on continuous groups, symmetry analysis provides a unified explanation for the seemingly diverse and ad hoc integration methods used to solve ordinary differential equations. At the present time, there is extensive literature on the subject and we refer the reader to the books by Bluman and Kumei [2],

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Olver [3] and Rogers and Ames [4]. For equations in $(1+1)$ dimensions, one seeks the invariance of a differential equation

$$\Omega(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots) = 0, \tag{1.1}$$

under the group of infinitesimal transformations

$$\begin{aligned}\bar{t} &= t + T(t, x, u)\epsilon + O(\epsilon^2),\\ \bar{x} &= x + X(t, x, u)\epsilon + O(\epsilon^2),\\ \bar{u} &= u + U(t, x, u)\epsilon + O(\epsilon^2).\end{aligned} \tag{1.2}$$

This leads to a set of determining equations for the infinitesimals T, X and U which, when solved, gives rise to the symmetries of (1.1). Once a symmetry is known for a differential equation, invariance of the solution leads to the invariant surface condition

$$Tu_t + Xu_x = U. \tag{1.3}$$

Solutions of (1.3) lead to a solution ansatz, which, substituted into Eq. (1.1) lead to a reduction of the original equation. A generalization of the so-called "classical method" of Lie was proposed by Bluman and Cole [5], which today is commonly referred to as the "nonclassical method." Their method seeks invariance of the original equation augmented with the invariant surface condition.

A particular class of partial differential equations that has benefited tremendously from this type of analysis are reaction–diffusion equations. These types of equations model a wide variety of physically interesting phenomena, and we refer the reader to Murray [6] for further discussion. The first account of the classical symmetry analysis of reaction–diffusion equation in one spatial dimension

$$u_t = [D(u)u_x]_x + Q(u), \tag{1.4}$$

was given by Dorodnitsyn [7]. In an exhaustive study, several forms of D and Q were given that provided a symmetry reduction of the original equation. This was subsequently followed by a nonclassical symmetry analysis of Eq. (1.4) by Arrigo et al. [8] and Clarkson and Mansfield [9] in the case of D constant, and then by Arrigo and Hill [10] in the case of exponential and power law type diffusivity. These papers all led to new exact solutions to the reaction–diffusion equation (1.4). For nonlinear convection–diffusion equation,

$$u_t = [D(u)u_x]_x + G'(u)u_x, \tag{1.5}$$

commonly referred to as Richard's equation, classical symmetry analysis in one space dimension was first given by Yung et al. [11] and Edwards [12] where again, several forms of D and G were given that gave rise to a symmetry reduction of the original equation. A generalized conditional symmetry method, independently developed by Fokas and Liu [13] and Zhdanov [14], was used by Qu [15] on the one-dimensional equation

$$u_t = [D(u)u_x]_x + P(u)u_x + Q(u), \tag{1.6}$$

an equation that encompasses both reaction and convection. Qu's analysis led to a classification of those types of equations that admit a generalized conditional symmetry in the case of exponential and power law type diffusivity. This was further extended to equations of the form

$$u_t = g(u)u_{xx} + f(u, u_x), \tag{1.7}$$

by Zhdanov and Andreitsev [16], who considered conditional symmetries of order 3, 4 and 5. For two- and three-dimensional reaction–diffusion equations of the form

$$u_t = \sum_{i=1}^{n} \big(D_i(u)u_{x_i}\big)_{x_i} + Q(u), \tag{1.8}$$

a classical symmetry analysis by Dorodnitsyn et al. [17] and by Galaktionov et al. [18] gave rise to the cases:

$$\text{(i)}\quad D_i(u) = D(u), \quad \text{for all } i, \tag{1.9a}$$

$$\text{(ii)}\quad D_i(u) \neq 0, \quad D_1/D_2 \neq \text{const}, \quad \text{for } n = 2, \tag{1.9b}$$

$$\text{(iii)}\quad D_i(u) \neq 0, \quad (D_1/D_2)^2 + (D_2/D_3)^2 + (D_3/D_1)^2 \neq 0, \quad \text{for } n = 3. \tag{1.9c}$$

Nonclassically, Eq. (1.8) without a source term (i.e., $Q(u) = 0$) was first considered by Arrigo et al. [19] in the case of $n = 2$, and with a source term by Goard and Broadbridge [20]. Gandarias and del Aguila [21] go on further to provide many reductions of (1.8) also in the case of $n = 2$ and $D = 1$. In the case of higher-dimensional diffusion equations with convection

$$u_t = \sum_{i=1}^{n} \big(D_i(u)u_{x_i}\big)_{x_i} + G(u)u_{x_n}, \tag{1.10}$$

a nonclassical analysis was first performed by Edwards and Broadbridge [22]. Further generalizations to systems of reaction–diffusion equations do not have as extensive a body of results as single equations. We note the work of Wiltshire [23] who investigated the case of coupled nonlinear diffusion equation without reaction, that of Baugh [24] and Nikitin and Wiltshire [25] who independently considered linear diffusion with reaction, and of Buchynchyk [26] who considered nonlinear diffusion with reaction.

The objective of the present paper is to obtain the symmetries of the linear diffusion equation

$$u_t = u_{xx} + u_{yy} + Q(u, u_x, u_y), \quad Q_{u_x} \neq 0,\ Q_{u_y} \neq 0, \tag{1.11}$$

and to determine those source terms Q that admit nontrivial symmetries. The paper is organized as follows. In Section 2, the determining equations of the classical symmetries of (1.11) are obtained and solved. Here several forms of the nonlinear source term are identified that admit a nontrivial symmetry. In Section 3, our analysis is then extended to the nonclassical symmetries of this same equation (1.11). Here we will show that the nonclassical method simply recovers the classical symmetries. An interesting fact is that the results obtained by Bindu et al. [27] do not appear within our results. In our final section we address this and show that it is necessary to split some of our determining equations in order to recover these results.

2. Classical symmetries

In this section we obtain and solve the determining equations for the classical symmetries of (1.11). If we let

$$\Delta = u_t - u_{xx} - u_{yy} - Q(u, u_x, u_y), \tag{2.1}$$

then invariance under the infinitesimal transformations

$$\begin{aligned}
\bar{t} &= t + \varepsilon T(t, x, y, u) + O(\varepsilon^2),\\
\bar{x} &= x + \varepsilon X(t, x, y, u) + O(\varepsilon^2),\\
\bar{y} &= y + \varepsilon Y(t, x, y, u) + O(\varepsilon^2),\\
\bar{u} &= u + \varepsilon U(t, x, y, u) + O(\varepsilon^2)
\end{aligned} \tag{2.2}$$

is conveniently written as

$$\Gamma^{(2)} \Delta\big|_{\Delta=0} = 0, \tag{2.3}$$

where the infinitesimal operator Γ is defined as

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + U\frac{\partial}{\partial U}, \tag{2.4}$$

and $\Gamma^{(1)}$ and $\Gamma^{(2)}$ are extensions to the operator Γ in (2.4), namely

$$\Gamma^{(1)} = \Gamma + U_{[t]}\frac{\partial}{\partial u_t} + U_{[x]}\frac{\partial}{\partial u_x} + U_{[y]}\frac{\partial}{\partial u_y}, \tag{2.5a}$$

$$\begin{aligned}
\Gamma^{(2)} = \Gamma^{(1)} &+ U_{[tt]}\frac{\partial}{\partial u_{tt}} + U_{[tx]}\frac{\partial}{\partial u_{tx}} + U_{[ty]}\frac{\partial}{\partial u_{ty}}\\
&+ U_{[xx]}\frac{\partial}{\partial u_{xx}} + U_{[xy]}\frac{\partial}{\partial u_{xy}} + U_{[yy]}\frac{\partial}{\partial u_{yy}}.
\end{aligned} \tag{2.5b}$$

The extended transformations are given by

$$U_{[t]} = D_t U - u_t D_t T - u_x D_t X - u_y D_t Y, \tag{2.6a}$$

$$U_{[x]} = D_x U - u_t D_x T - u_x D_x X - u_y D_x Y, \tag{2.6b}$$

$$U_{[y]} = D_y U - u_t D_y T - u_x D_y X - u_y D_y Y, \tag{2.6c}$$

and

$$U_{[tt]} = D_t U_{[t]} - u_{tt} D_t T - u_{tx} D_t X - u_{ty} D_t Y, \tag{2.7a}$$

$$U_{[tx]} = D_x U_{[t]} - u_{tt} D_x T - u_{tx} D_x X - u_{ty} D_x Y, \tag{2.7b}$$

$$U_{[ty]} = D_y U_{[t]} - u_{tt} D_y T - u_{tx} D_y X - u_{ty} D_y Y, \tag{2.7c}$$

$$U_{[xx]} = D_x U_{[x]} - u_{tx} D_x T - u_{xx} D_x X - u_{xy} D_x Y, \tag{2.7d}$$

$$U_{[xy]} = D_x U_{[y]} - u_{ty} D_x T - u_{xy} D_x X - u_{yy} D_x Y, \tag{2.7e}$$

$$U_{[yy]} = D_y U_{[y]} - u_{ty} D_y T - u_{xy} D_y X - u_{yy} D_y Y, \tag{2.7f}$$

where the total differential operators D_t , D_x and D_y are given by

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + u_{ty}\frac{\partial}{\partial u_y} + u_{ttt}\frac{\partial}{\partial u_{tt}}\cdots, \tag{2.8a}$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + u_{ttx}\frac{\partial}{\partial u_{tt}}\cdots, \tag{2.8b}$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{ty}\frac{\partial}{\partial u_t} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + u_{tty}\frac{\partial}{\partial u_{tt}}\cdots. \tag{2.8c}$$

Applying Lie's invariance condition (2.3) gives rise to the following determining equations for T, X, Y and U:

$$T_x + pT_u = 0, \tag{2.9a}$$
$$T_y + qT_u = 0, \tag{2.9b}$$
$$X_x - Y_y + pX_u - qY_u = 0, \tag{2.9c}$$
$$Y_x + X_y + pY_u + qX_u = 0, \tag{2.9d}$$
$$\begin{aligned}&(T_x + pT_u)Q_p + (T_y + qT_u)Q_q - T_u Q + 2X_x - T_t\\&\quad + T_{xx} + T_{yy} + 2(X_u + T_{xu})p + 2(Y_u + T_{yu})q + \left(p^2 + q^2\right)T_{uu} = 0,\end{aligned} \tag{2.9e}$$
$$\begin{aligned}&U_t - U_{xx} - U_{yy} - 2pU_{xu} - 2qU_{yu} - \left(p^2 + q^2\right)U_{uu}\\&\quad - pX_t + pX_{xx} + pX_{yy} + 2p^2X_{xu} + 2pqX_{yu} + p\left(p^2 + q^2\right)X_{uu}\\&\quad - qY_t + qY_{xx} + qY_{yy} + 2pqY_{xu} + 2q^2Y_{yu} + q\left(p^2 + q^2\right)Y_{uu}\\&\quad + (U_u - 2X_x - 3pX_u - qY_u)Q + (-U_x - pU_u + pX_x + qY_x + p^2X_u + pqY_u)Q_p\\&\quad - UQ_u + \left(-U_y - qU_u + pX_y + qY_y + pqX_u + q^2Y_u\right)Q_q = 0,\end{aligned} \tag{2.9f}$$

where we have adopted the usual notation that subscripts refer to partial differentiation and that $u_x = p$ and $u_y = q$. Since T, X and Y are independent of p and q, we see from (2.9a)–(2.9e) that

$$T_x = 0, \qquad T_y = 0, \qquad T_u = 0, \qquad X_u = 0, \qquad Y_u = 0, \tag{2.10}$$

which gives that

$$T = T(t), \qquad X = X(t, x, y), \qquad Y = Y(t, x, y), \tag{2.11}$$

where T, X and Y satisfy

$$T_t - 2X_x = 0, \qquad X_x - Y_y = 0, \qquad Y_x + X_y = 0. \tag{2.12}$$

With these simplifications, (2.9f) becomes

$$\begin{aligned}&U_t - U_{xx} - U_{yy} - \left(p^2 + q^2\right)U_{uu} + (U_u - 2X_x)Q - UQ_u\\&\quad + \left((X_x - U_u)p + Y_xq - U_x\right)Q_p + \left(X_yp + (Y_y - U_u)q - U_y\right)Q_q\\&\quad - (2U_{xu} + X_t)p - (2U_{yu} + Y_t)q = 0.\end{aligned} \tag{2.13}$$

We further find from (2.12) that

$$X = \frac{1}{2}T'(t)x + A(t)y + B(t), \qquad Y = -A(t)x + \frac{1}{2}T'(t)y + C(t), \tag{2.14}$$

where A, B and C are arbitrary functions of t. Substitution of $T = T(t)$ and (2.14) into (2.13) gives

$$\begin{aligned}&U_t - U_{xx} - U_{yy} - \left(p^2 + q^2\right)U_{uu} + (U_u - T')Q - UQ_u\\&\quad + \left(\left(\frac{1}{2}T' - U_u\right)p - Aq - U_x\right)Q_p + \left(Ap + \left(\frac{1}{2}T' - U_u\right)q - U_y\right)Q_q\\&\quad - \left(2U_{xu} + \frac{1}{2}T''x + A'y + B'\right)p - \left(2U_{yu} - A'x + \frac{1}{2}T''y + C'\right)q = 0.\end{aligned} \tag{2.15}$$

Our goal now is to determine forms of $Q(u, p, q)$ and corresponding functions $T(t)$, $A(t)$, $B(t)$, $C(t)$ and $U(t, x, y, u)$ such that (2.15) is satisfied. This leads to two special cases: $U = 0$ and $U \neq 0$. Each will be considered separately noting that arbitrary constants are denoted by c_i, $i = 0, 1, 2, \ldots$.

2.1. $U(t,x,y,u)=0$

If we set $U=0$, then Eq. (2.15) becomes

$$\left(\frac{1}{2}T'p - Aq\right)Q_p + \left(Ap + \frac{1}{2}T'q\right)Q_q - T'Q - \left(\frac{1}{2}T''x + A'y + B'\right)p - \left(-A'x + \frac{1}{2}T''y + C'\right)q = 0. \tag{2.16}$$

Differentiating (2.16) with respect to xp and xq (or yp and yq) gives $T''=0$ and $A'=0$ from which we deduce that

$$T(t) = 2c_1t + c_0, \qquad A(t) = c_2.$$

This, in turn, gives (2.16) as

$$(c_1p - c_2q)Q_p + (c_2p + c_1q)Q_q - 2c_1Q - pB'(t) - qC'(t) = 0.$$

As Q is independent of t, this forces B' and C' to be constant giving

$$B(t) = c_3t + c_4, \qquad C(t) = c_5t + c_6.$$

Thus, we finally have a single equation for Q, namely

$$(c_1p - c_2q)Q_p + (c_2p + c_1q)Q_q - 2c_1Q - c_3p - c_5q = 0. \tag{2.17}$$

The solution of (2.17) depends on whether the constants c_1 and c_2 are zero, noting that in the case where $c_1 = c_2 = 0$, then $c_3 = c_5 = 0$ which lead to translational symmetries in t, x and y for arbitrary Q.

Case (i): $c_1 = 0$, $c_2 \neq 0$.
From (2.17) we have

$$-c_2qQ_p + c_2pQ_q - c_3p - c_5q = 0,$$

which has the solution

$$Q = F\big(u, p^2 + q^2\big) - \frac{c_5}{c_2}p + \frac{c_3}{c_2}q,$$

where F is an arbitrary function. The associated infinitesimals for this particular source term are

$$T = c_0, \qquad X = c_2y + c_3t + c_4, \qquad Y = -c_2x + c_5t + c_6, \qquad U = 0.$$

Case (ii): $c_1 \neq 0$, $c_2 = 0$.
From (2.17) we have

$$c_1pQ_p + c_1qQ_q - 2c_1Q - c_3p - c_5q = 0,$$

which has the solution

$$Q = p^2F\left(u, \frac{q}{p}\right) - \frac{c_3}{c_1}p - \frac{c_5}{c_1}q,$$

where F is an arbitrary function. The associated infinitesimals for this particular source term are

$$T = 2c_1t + c_0, \qquad X = c_1x + c_3t + c_4, \qquad Y = c_1y + c_5t + c_6, \qquad U = 0.$$

Case (iii): $c_1 \neq 0$, $c_2 \neq 0$.

If we switch to polar coordinates $p = r\cos\theta$, $q = r\sin\theta$, Eq. (2.17) becomes

$$c_1 r Q_r + c_2 Q_\theta - 2c_1 Q - c_3 r\cos\theta - c_5 r\sin\theta = 0,$$

which has the solution

$$Q = r^2 F\left(u, \theta - \frac{c_2}{c_1}\ln r\right) - \frac{c_1c_3 + c_2c_5}{c_1^2 + c_2^2} r\cos\theta - \frac{c_1c_5 - c_2c_3}{c_1^2 + c_2^2} r\sin\theta$$

or, in terms of the original variables

$$Q = \left(p^2 + q^2\right) F\left(u, \tan^{-1}\frac{q}{p} - \frac{c_2}{2c_1}\ln p^2 + q^2\right) - \frac{c_1c_3 + c_2c_5}{c_1^2 + c_2^2} p - \frac{c_1c_5 - c_2c_3}{c_1^2 + c_2^2} q,$$

where F is an arbitrary function. The associated infinitesimals for this particular source term are

$$T = 2c_1 t + c_0, \qquad X = c_1 x + c_2 y + c_3 t + c_4,$$
$$Y = -c_2 x + c_1 y + c_5 t + c_6, \qquad U = 0.$$

2.2. $U(t, x, y, u) \neq 0$

Dividing Eq. (2.15) by $-U$ and re-grouping gives

$$\begin{aligned} &Q_u + \left(\frac{(U_u - \frac{1}{2}T')}{U} p + \frac{A}{U} q + \frac{U_x}{U}\right) Q_p + \left(-\frac{A}{U} p + \frac{(U_u - \frac{1}{2}T')}{U} q + \frac{U_y}{U}\right) Q_q \\ &\quad + \frac{(T' - U_u)}{U} Q + \left(p^2 + q^2\right)\frac{U_{uu}}{U} + \frac{(2U_{xu} + \frac{1}{2}T''x + A'y + B')}{U} p \\ &\quad + \frac{(2U_{yu} - A'x + \frac{1}{2}T''y + C')}{U} q + \frac{U_{xx} + U_{yy} - U_t}{U} = 0. \end{aligned} \tag{2.18}$$

As Q is only a function of p, q and u, then each coefficient of (2.18) can be at most a function of u. If we let

$$\begin{aligned} &\frac{U_u - \frac{1}{2}T'}{U} = \lambda_1(u), \qquad \frac{A}{U} = \lambda_2(u), \qquad \frac{U_x}{U} = \lambda_3(u), \qquad \frac{U_y}{U} = \lambda_4(u), \\ &\frac{T' - U_u}{U} = \lambda_5(u), \qquad \frac{U_{uu}}{U} = \lambda_6(u), \qquad \frac{2U_{xu} + \frac{1}{2}T''x + A'y + B'}{U} = \lambda_7(u), \\ &\frac{2U_{yu} - A'x + \frac{1}{2}T''y + C'}{U} = \lambda_8(u), \qquad \frac{U_{xx} + U_{yy} - U_t}{U} = \lambda_9(u), \end{aligned} \tag{2.19}$$

then (2.18) becomes

$$\begin{aligned} &Q_u + (\lambda_1 p + \lambda_2 q + \lambda_3) Q_p + (-\lambda_2 p + \lambda_1 q + \lambda_4) Q_q + \lambda_5 Q + \lambda_6\left(p^2 + q^2\right) \\ &\quad + \lambda_7 p + \lambda_8 q + \lambda_9 = 0, \end{aligned} \tag{2.20}$$

where λ_1–λ_9 are to be determined. Our investigation of (2.19) leads us to consider three cases:

(i) $\lambda_2 \neq 0$,

(ii) $\lambda_2 = 0$, $\quad \lambda_1 + \lambda_5 \neq 0$,

(iii) $\lambda_2 = 0$, $\quad \lambda_1 + \lambda_5 = 0$.

Each case will be considered separately.

Case (i): $\lambda_2 \neq 0$.

If $\lambda_2 \neq 0$, then the second equation of (2.19) gives

$$U = A(t)k(u), \tag{2.21}$$

where $k(u) = 1/\lambda_2(u)$. With this assignment, we further deduce from the seventh and eighth equations of (2.19) that $T'' = 0$ and $A' = 0$ since U is independent of both x and y. This gives

$$T = 2c_1t + c_0, \qquad A = c_2, \tag{2.22}$$

which implies that $U = c_2k(u)$. Since the only time dependence in (2.19) is through B' and C', this forces these to be constant. Thus,

$$B = c_3t + c_4, \qquad C = c_5t + c_6. \tag{2.23}$$

This gives, from (2.14), (2.21)–(2.23) the infinitesimals

$$\begin{aligned} &T = 2c_1t + c_0, \qquad X = c_1x + c_2y + c_3t + c_4, \\ &Y = -c_2x + c_1y + c_5t + c_6, \qquad U = c_2k(u). \end{aligned} \tag{2.24}$$

Therefore, in this case (2.19) reduces to

$$\begin{aligned} &\lambda_1 = \frac{c_2k'(u) - c_1}{c_2k(u)}, \qquad \lambda_2 = \frac{1}{k(u)}, \qquad \lambda_3 = 0, \qquad \lambda_4 = 0, \\ &\lambda_5 = \frac{2c_1 - c_2k'(u)}{c_2k(u)}, \qquad \lambda_6 = \frac{k''}{k}, \qquad \lambda_7 = \frac{c_3}{c_2k(u)}, \qquad \lambda_8 = \frac{c_5}{c_2k(u)}, \qquad \lambda_9(u) = 0, \end{aligned} \tag{2.25}$$

and thus it follows from (2.20) and (2.25) that any Q satisfying

$$\begin{aligned} &c_2kQ_u + \left[(c_2k' - c_1)p + c_2q\right]Q_p + \left[-c_2p + (c_2k' - c_1)q\right]Q_q \\ &\quad + (2c_1 - c_2k')Q + c_2k''\left(p^2 + q^2\right) + c_3p + c_5q = 0, \end{aligned} \tag{2.26}$$

is left invariant under (1.2) with the infinitesimals as given in (2.24) where $k(u)$ is any arbitrary function of u. The solution of Eq. (2.26) is given by

$$\begin{aligned} Q = {} & k(u)e^{-2\frac{c_1}{c_2}\int \frac{du}{k}} F\left(\frac{1}{2}\ln p^2 + q^2 - \ln k + \frac{c_1}{c_2}\int \frac{du}{k}, \tan^{-1}\frac{q}{p} + \int \frac{du}{k}\right) \\ & - \frac{k'}{k}\left(p^2 + q^2\right) - \frac{c_1c_3 + c_2c_5}{c_1^2 + c_2^2}p + \frac{c_1c_5 - c_2c_3}{c_1^2 + c_2^2}q. \end{aligned}$$

Case (ii): $\lambda_2 = 0$, $\lambda_1 + \lambda_5 \neq 0$.

With $\lambda_2 = 0$, from (2.19) we have that $A = 0$. Since $\lambda_1 + \lambda_5 \neq 0$, if we add the first and fifth equations of (2.19) then

$$\frac{\frac{1}{2}T'}{U} = \lambda_1 + \lambda_5,$$

and, if we set

$$\lambda_1 + \lambda_5 = \frac{1}{2k(u)},$$

then

$$U = T'k(u).$$

From the seventh and eighth equations of (2.19) we see that

$$\frac{\frac{1}{2}T''x + B'}{T'k} = \lambda_7, \qquad \frac{\frac{1}{2}T''y + C'}{T'k} = \lambda_8,$$

from which we deduce that $T'' = 0$, and then $B'' = 0$ and $C'' = 0$. Integrating gives

$$T = 2c_1 t + c_0, \qquad B = c_2 t + c_3, \qquad C = c_4 t + c_5. \tag{2.27}$$

Thus, (2.14) with T, B and C given in (2.27) gives the infinitesimals as

$$\begin{aligned} &T = 2c_1 t + c_0, \qquad X = c_1 x + c_2 t + c_3, \\ &Y = c_1 y + c_4 t + c_5, \qquad U = 2c_1 k(u). \end{aligned} \tag{2.28}$$

From (2.19) we see that

$$\lambda_1 = \frac{k' - \frac{1}{2}}{k}, \qquad \lambda_5 = \frac{1 - k'}{k}, \qquad \lambda_6 = \frac{k''}{k}, \qquad \lambda_7 = \frac{c_2}{2c_1 k}, \qquad \lambda_8 = \frac{c_4}{2c_1 k} \tag{2.29}$$

with all other λ's zero. Thus, it follows from (2.20) and (2.29) that any Q satisfying

$$\begin{aligned} &kQ_u + \left(k' - \frac{1}{2}\right)pQ_p + \left(k' - \frac{1}{2}\right)qQ_q + (1 - k')Q + k''\left(p^2 + q^2\right) \\ &\quad + \frac{c_2}{2c_1}p + \frac{c_4}{2c_1}q = 0 \end{aligned} \tag{2.30}$$

is left invariant under (1.2) with the infinitesimals as given in (2.28) where $k(u)$ is any arbitrary function of u. The solution of (2.30) is given by

$$\begin{aligned} Q = {} & k(u)e^{-\int \frac{du}{k}} F\left(\frac{1}{2}\ln p^2 + q^2 - \ln k + \frac{1}{2}\int \frac{du}{k}, \tan^{-1}\frac{q}{p}\right) \\ & - \frac{k'}{k}\left(p^2 + q^2\right) - \frac{c_2}{c_1}p - \frac{c_4}{c_1}q. \end{aligned}$$

Case (iii): $\lambda_2 = 0$, $\lambda_1 + \lambda_5 = 0$.

If $\lambda_2 = 0$, then from (2.19) $A = 0$. Since $\lambda_5 = -\lambda_1$, then adding the first and fifth equations in (2.19) gives $T' = 0$ from which we obtain $T = c_1$. Integrating the first equation of (2.19) gives

$$U = f(t, x, y)k(u), \tag{2.31}$$

where f is an arbitrary function of x, y and t and

$$k(u) = e^{\int \lambda_1(u)\,du}.$$

Substituting (2.31) into the third and fourth equations of (2.19) gives

$$\frac{f_x}{f} = \lambda_3(u), \qquad \frac{f_y}{f} = \lambda_4(u),$$

from which we deduce that λ_3 and λ_4 are constant and

$$f = f_0(t)e^{\lambda_3 x + \lambda_4 y}, \tag{2.32}$$

with f_0 an arbitrary function of t. Substituting (2.31) with (2.32) into the ninth equation of (2.19) gives

$$\frac{(\lambda_3^2+\lambda_4^2)f_0 - f_0'}{f_0} = \lambda_9(u), \tag{2.33}$$

showing that λ_9 is also constant. Integrating (2.33) gives f_0 as

$$f_0 = U_0 e^{(\lambda_3^2+\lambda_4^2-\lambda_9)t} \tag{2.34}$$

where U_0 is a constant. Thus, combining (2.31), (2.32), and (2.34) gives

$$U = U_0 e^{\lambda_3 x+\lambda_4 y+(\lambda_3^2+\lambda_4^2-\lambda_9)t} k(u). \tag{2.35}$$

Substituting (2.35) into the seventh and eighth equations of (2.19) and simplifying gives

$$2\lambda_3 \frac{k'(u)}{k(u)} + \frac{B'(t)}{U_0 k(u)} e^{-\lambda_3 x-\lambda_4 y-(\lambda_3^2+\lambda_4^2-\lambda_9)t} = \lambda_7(u), \tag{2.36a}$$

$$2\lambda_4 \frac{k'(u)}{k(u)} + \frac{C'(t)}{U_0 k(u)} e^{-\lambda_3 x-\lambda_4 y-(\lambda_3^2+\lambda_4^2-\lambda_9)t} = \lambda_8(u). \tag{2.36b}$$

As (2.36) must be independent of both x and y this gives rise to two cases: $\lambda_3 = \lambda_4 = 0$ or $B' = C' = 0$.

Subcase (i): $\lambda_3 = \lambda_4 = 0$.

In this case, we conclude from (2.36)

$$B'(t) = b_1 U_0 e^{-\lambda_9 t}, \qquad C'(t) = c_1 U_0 e^{-\lambda_9 t}, \tag{2.37}$$

with b_1 and c_1 arbitrary constants and

$$\lambda_7 = \frac{b_1}{k(u)}, \qquad \lambda_8 = \frac{c_1}{k(u)}.$$

If $\lambda_9 = 0$, then integrating (2.37) gives

$$B = b_1 U_0 t + b_0, \qquad C = c_1 U_0 t + c_0,$$

where b_0 and c_0 are arbitrary constants. This, in turn, leads to the infinitesimals

$$T = c_1, \qquad X = b_1 U_0 t + b_0, \qquad Y = c_1 U_0 t + c_0, \qquad U = U_0 k(u). \tag{2.38}$$

If $\lambda_9 \neq 0$, then integrating (2.37) gives

$$B = -\frac{b_1 U_0}{\lambda_9} e^{-\lambda_9 t} + b_0, \qquad C = -\frac{c_1 U_0}{\lambda_9} e^{-\lambda_9 t} + c_0,$$

where b_0 and c_0 are arbitrary constants. This, in turn, leads to the infinitesimals

$$T = c_1, \qquad X = -\frac{b_1 U_0}{\lambda_9} e^{-\lambda_9 t} + b_0, \qquad Y = -\frac{c_1 U_0}{\lambda_9} e^{-\lambda_9 t} + c_0, \qquad U = U_0 e^{-\lambda_9 t} k(u). \tag{2.39}$$

These infinitesimals ((2.38) and (2.39)) apply for source terms that satisfy

$$k Q_u + k' p Q_p + k' q Q_q - k' Q + k''(p^2+q^2) + b_1 p + c_1 q + \lambda_9 k = 0, \tag{2.40}$$

depending whether λ_9 is zero or not. The solution of (2.40) is

$$Q = kF\left(\frac{p}{k}, \frac{q}{k}\right) - \frac{k'}{k}(p^2+q^2) - (b_1 p + c_1 q + \lambda_9 k) \int \frac{du}{k}.$$

Subcase (ii): $B' = C' = 0$.

In this case $B = b_0$ and $C = c_0$ are arbitrary constants and from (2.36), $\lambda_7 = 2\lambda_3 k'/k$ and $\lambda_8 = 2\lambda_4 k'/k$. This, in turn, leads to the infinitesimals

$$T = c_1, \qquad X = b_0, \qquad Y = c_0, \qquad U = U_0 e^{\lambda_3 x + \lambda_4 y + (\lambda_3^2 + \lambda_4^2 - \lambda_9)t} k(u), \tag{2.41}$$

which apply for source terms that satisfy

$$\begin{aligned} &kQ_u + (k'p + \lambda_3 k)Q_p + (k'q + \lambda_4 k)Q_q - k'Q + k''\big(p^2 + q^2\big) \\ &\quad + 2\lambda_3 k' p + 2\lambda_4 k' q + \lambda_9 k = 0. \end{aligned} \tag{2.42}$$

The solution of Eq. (2.42) is given by

$$Q = kF\left(\frac{p}{k} - \lambda_3 \int \frac{du}{k}, \frac{q}{k} - \lambda_4 \int \frac{du}{k}\right) - \frac{k'}{k}\big(p^2 + q^2\big) - \lambda_9 k \int \frac{du}{k}.$$

3. Nonclassical symmetries

For the nonclassical method, we seek invariance of both the original equation and its invariant surface condition. This can also be conveniently written as

$$\Gamma^{(2)} \Delta_1 \big|_{\Delta_1 = 0,\, \Delta_2 = 0} = 0, \tag{3.1}$$

where Δ_1 and Δ_2 are defined as

$$\Delta_1 = u_t - u_{xx} - u_{yy} - Q(u, u_x, u_y), \qquad \Delta_2 = u_t + Xu_x - U, \tag{3.2}$$

noting that we can set $T = 1$ without of loss of generality provided that $T = 0$. In (3.1), the infinitesimal operator Γ, its first and second extensions and extended transformations can all be found in the previous section, (2.4)–(2.7), and we refer the reader there. Applying (3.1) gives rise to the following determining equations for X, Y and U:

$$X_x - Y_y + pX_u - qY_u = 0, \tag{3.3a}$$

$$Y_x + X_y + pY_u + qX_u = 0, \tag{3.3b}$$

$$\begin{aligned} &U_t - U_{xx} - U_{yy} - 2pU_{xu} - 2qU_{yu} - (p^2 + q^2)U_{uu} \\ &\quad - pX_t + pX_{xx} + pX_{yy} + 2p^2 X_{xu} + 2pqX_{yu} + p\big(p^2 + q^2\big)X_{uu} \\ &\quad - qY_t + qY_{xx} + qY_{yy} + 2pqY_{xu} + 2q^2 Y_{yu} + q\big(p^2 + q^2\big)Y_{uu} \\ &\quad + 2(U - pX - qY)(X_x + pX_u) \\ &\quad + (U_u - 2X_x - 3pX_u - qY_u)Q + \big(-U_x - pU_u + pX_x + qY_x + p^2 X_u + pqY_u\big)Q_p \\ &\quad - UQ_u + \big(-U_y - qU_u + pX_y + qY_y + pqX_u + q^2 Y_u\big)Q_q = 0. \end{aligned} \tag{3.3c}$$

From (3.3a) and (3.3b) we find that

$$X = X(t, x, y), \qquad Y = X(t, x, y), \tag{3.4}$$

where

$$X_x - Y_y = 0, \qquad Y_x + X_y = 0, \tag{3.5}$$

and from (3.3c)

$$\begin{aligned}&U_t - U_{xx} - U_{yy} + 2X_xU - \left(p^2+q^2\right)U_{uu} + (U_u - 2X_x)Q - UQ_u\\&\quad + \big((X_x - U_u)p + Y_xq - U_x\big)Q_p + \big(X_yp + (Y_y - U_u)q - U_y\big)Q_q\\&\quad - (2U_{xu} + X_t + 2XX_x)p - (2U_{yu} + Y_t + 2YX_x)q = 0.\end{aligned} \tag{3.6}$$

If we let $X \to X/T$, $Y \to Y/T$ and $U \to U/T$ where $T = T(t)$, then (3.6) becomes

$$\begin{aligned}&U_t - U_{xx} - U_{yy} + \frac{U}{T}(2X_x - T_t) - \left(p^2+q^2\right)U_{uu} + (U_u - 2X_x)Q - UQ_u\\&\quad + \big((X_x - U_u)p + Y_xq - U_x\big)Q_p + \big(X_yp + (Y_y - U_u)q - U_y\big)Q_q\\&\quad - \left(2U_{xu} + X_t + \frac{X}{T}(2X_x - T_t)\right)p - \left(2U_{yu} + Y_t + \frac{Y}{T}(2X_x - T_t)\right)q = 0.\end{aligned} \tag{3.7}$$

Comparing (3.7) with (2.13) shows that they are identical if $2X_x - T_t = 0$. As with the classical symmetries where several cases were considered ($U = 0$ and $U \neq 0$), we will also consider these cases separately where we will establish that $2X_x - T_t = 0$.

3.1. $U(t, x, y, u) = 0$

If we set $U = 0$, then Eq. (3.7) becomes

$$\begin{aligned}&-2X_xQ + (X_xp + Y_xq)Q_p + (X_yp + Y_yq)Q_q\\&\quad - \left(X_t + \frac{X}{T}(2X_x - T_t)\right)p - \left(Y_t + \frac{Y}{T}(2X_x - T_t)\right)q = 0.\end{aligned} \tag{3.8}$$

Differentiating (3.8) twice with respect to x and y and adding gives, using $X_{xx} + X_{yy} = 0$ and $Y_{xx} + Y_{yy} = 0$,

$$\big(X_x(2X_x - T_t)_x + X_y(2X_x - T_t)_y\big)p + \big(Y_x(2X_x - T_t)_x + Y_y(2X_x - T_t)_y\big)q = 0,$$

which gives

$$X_x(2X_x - T_t)_x + X_y(2X_x - T_t)_y = 0, \tag{3.9a}$$

$$Y_x(2X_x - T_t)_x + Y_y(2X_x - T_t)_y = 0. \tag{3.9b}$$

From (3.9) and (3.5) we find that either

$$X_x = 0, \qquad X_y = 0, \qquad Y_x = 0, \qquad Y_y = 0, \tag{3.10}$$

or

$$(2X_x - T_t)_x = 0, \qquad (2X_x - T_t)_y = 0. \tag{3.11}$$

We choose the latter as it is clearly more general. Integrating (3.11) gives

$$2X_x - T_t = k(t) \tag{3.12}$$

for some arbitrary function k but this can be absorbed into T without loss of generality thus establishing that $2X_x - T_t = 0$.

3.2. $U(t,x,y,u)\neq 0$

Dividing Eq. (3.7) by $-U$ and re-grouping gives

$$\begin{aligned}
&Q_u+\left(\frac{(U_u-X_x)}{U}p-\frac{Y_x}{U}q+\frac{U_x}{U}\right)Q_p+\left(-\frac{X_y}{U}p+\frac{(U_u-Y_y)}{U}q+\frac{U_y}{U}\right)Q_q\\
&\quad+\frac{(2X_x-U_u)}{U}Q+\left(p^2+q^2\right)\frac{U_{uu}}{U}+\frac{1}{U}\left(2U_{xu}+X_t+\frac{X}{T}(2X_x-T_t)\right)p\\
&\quad+\frac{1}{U}\left(2U_{yu}+Y_t+\frac{Y}{T}(2X_x-T_t)\right)q+\frac{1}{U}\left(U_{xx}+U_{yy}-U_t-\frac{U}{T}(2X_x-T_t)\right)=0.
\end{aligned}\tag{3.13}$$

As seen in the previous section using the classical method, as Q is only a function of p, q and u, then each coefficient of (3.13) can be at most a function of u. If we let

$$\begin{aligned}
&\frac{U_u-X_x}{U}=\lambda_1(u),\qquad -\frac{Y_x}{U}=\lambda_2(u),\qquad \frac{U_x}{U}=\lambda_3(u),\qquad \frac{U_y}{U}=\lambda_4(u),\\
&\frac{2X_x-U_u}{U}=\lambda_5(u),\qquad \frac{U_{uu}}{U}=\lambda_6(u),\qquad \frac{1}{U}\left(2U_{xu}+X_t+\frac{X}{T}(2X_x-T_t)\right)=\lambda_7(u),\\
&\frac{1}{U}\left(2U_{yu}+Y_t+\frac{Y}{T}(2X_x-T_t)\right)=\lambda_8(u),\\
&\frac{1}{U}\left(U_{xx}+U_{yy}-U_t-\frac{U}{T}(2X_x-T_t)\right)=\lambda_9(u),
\end{aligned}\tag{3.14}$$

then (3.13) becomes

$$\begin{aligned}
&Q_u+(\lambda_1p+\lambda_2q+\lambda_3)Q_p+(-\lambda_2p+\lambda_1q+\lambda_4)Q_q+\lambda_5Q+\lambda_6\left(p^2+q^2\right)\\
&\quad+\lambda_7p+\lambda_8q+\lambda_9=0,
\end{aligned}\tag{3.15}$$

exactly the same as (2.20) derived using the classical method. As the investigation there leads to considering three cases: (i) $\lambda_2\neq 0$, (ii) $\lambda_2=0$, $\lambda_1+\lambda_5\neq 0$, (iii) $\lambda_2=0$, $\lambda_1+\lambda_5=0$, we do the same here.

Case (i): $\lambda_2\neq 0$.

If $\lambda_2\neq 0$, then the second equation of (3.14) gives

$$U=Y_xk(u),\tag{3.16}$$

where $k(u)=1/\lambda_2(u)$. Since Y is Laplacian in x and y, then from (3.16), so is U. From the seventh and eighth equations of (3.14), we have

$$2U_{xu}+X_t+\frac{X}{T}(2X_x-T_t)=\lambda_7(u)U,\tag{3.17a}$$

$$2U_{yu}+Y_t+\frac{Y}{T}(2X_x-T_t)=\lambda_8(u)U.\tag{3.17b}$$

Differentiating both (3.17a) and (3.17b) with respect to x and y twice and adding gives (3.9). As we deduced there that $2X_x-T_t=0$, then so is the case here.

Case (ii): $\lambda_2 = 0$, $\lambda_1 + \lambda_5 \neq 0$.

If $\lambda_1 + \lambda_5 \neq 0$, then adding the first and fifth equations of (3.14) gives

$$U = X_x k(u), \tag{3.18}$$

where $k(u) = 1/(\lambda_1(u) + \lambda_5(u))$. Since X is Laplacian in x and y, then again, so is U. As we recognize that this case is the same as the previous case, we therefore deduce by the arguments presented in case (i) that $2X_x - T_t = 0$.

Case (iii): $\lambda_2 = 0$, $\lambda_1 + \lambda_5 = 0$.

If $\lambda_2 = 0$ and $\lambda_1 + \lambda_5 = 0$, then the second equation of (3.14), first and fifth of (3.14) and (3.5) give

$$X_x = 0, \qquad X_y = 0, \qquad Y_x = 0, \qquad Y_y = 0, \tag{3.19}$$

so $X = X(t)$ and $Y = Y(t)$ only. Furthermore, comparing (2.13) and (3.6) shows that they are identical! Thus, the source terms from each method will be identical. In the classical method $T = c_1$. Since X, Y and U in the classical method can be scaled by c_1 and with this scaling (2.13) remains invariant, we therefore can set $c_1 = 1$ without loss of generality. We thus conclude that in this case the nonclassical method recovers the classical method.

4. Conclusion

In this paper we have considered the symmetry analysis of a $(2+1)$-dimensional diffusion equation with a nonlinear source term that involves both the dependent variable and its first derivatives. In the first part of the paper, the classical method was used to classify those source terms that admit a nontrivial symmetry. The second part of the paper used the nonclassical method to show there were no nonclassical symmetries—those symmetries that cannot be obtained classically. It is interesting to note that the case where Q is of the form

$$Q = \frac{2}{1-u}\left(u_x^2 + u_y^2\right) + u(1-u) \tag{4.1}$$

does not appear among our results. This is a source term for (1.11) that Bindu et al. [27] showed admitted infinitesimal transformations with the infinitesimals

$$T = c_0, \qquad X = c_1 y + c_2, \qquad Y = -c_1 x + c_3, \qquad U = f(t,x,y)(1-u)^2 \tag{4.2}$$

where $f_t = f_{xx} + f_{yy} + f$. They further showed that under the transformation $u = 1 - 1/v$, the original PDE linearized. It is therefore natural to ask why this result is not contained within ours. The reason lies in our focus on Eq. (2.18) for Q. We have assumed that all the terms in this equation must appear. In order to obtain the results of Bindu et al., it is necessary to split Eq. (2.18). We admit that in doing so, the source term might not be a general as those obtained in Section 2 but it is possible (and we will demonstrate this) that a larger symmetry group can be obtained. We now re-examine (2.15) with first a re-organization

$$\begin{aligned} &A'(xq - yp) - B'p - C'q + \frac{1}{2}T'(pQ_p + qQ_q - 2Q) - \frac{1}{2}T''(xp + yq) \\ &\quad - A(qQ_p - pQ_q) - U_x Q_p - 2U_{xu} p - U_y Q_q - 2U_{yu} q + U_t - U_{xx} - U_{yy} \\ &\quad - UQ_u - U_u(pQ_p + qQ_q - Q) - \left(p^2 + q^2\right)U_{uu} = 0. \end{aligned} \tag{4.3}$$

We first set the coefficients of the first four terms to zero

$$A' = B' = C' = T' = 0, \tag{4.4}$$

and then split the remaining equation according to

$$U_x Q_p + 2pU_{xu} = 0, \qquad U_y Q_q + 2qU_{yu} = 0, \tag{4.5a}$$

$$qQ_p - pQ_q = 0, \qquad U_t - U_{xx} - U_{yy} = k_1 U, \tag{4.5b}$$

$$UQ_u + U_u(pQ_p + qQ_q - Q) + (p^2 + q^2)U_{uu} = k_1 U, \tag{4.5c}$$

for some arbitrary constant k_1. In the case of $k_1 = 1$, (4.5) is identically satisfied by (4.2) if we choose the source term (4.1). We further note that if we solve (4.4) and (4.5) in general, we obtain

$$T = c_0, \qquad X = c_1 y + c_2, \qquad Y = -c_1 x + c_3, \qquad U = \frac{f(t,x,y)}{S'(u)},$$
$$Q = \frac{S''(u)}{S'(u)}(p^2 + q^2) + \frac{k_1 S(u) + k_2}{S'(u)}, \tag{4.6}$$

where k_2 is another arbitrary constant, $S(u)$ an arbitrary function and $f(t,x,y)$ a function satisfying $f_t = f_{xx} + f_{yy} + k_1 f$, thus generalizing the result of Bindu et al. [27]. We further note that for this source term (4.6), the substitution $v = S(u)$ linearizes (1.11).

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